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GENERALIZED EQUATION OF MOISTURE DIFFUSIVITY AND ITS SOLUTION

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The generalized equation of moisture diffusivity in capillary-porous bodies is examined, and solutions are given for different boundary conditions.

In 1962 A. V. Luikov proposed and substantiated the hypothesis that the propagation of heat and mass in heat and moisture transfer processes in capillary-porous bodies proceed at a finite rate. A similar idea was advanced by P. Vernott and C. Cattaneo.

The author of the hypothesis of a finite rate of heat and mass transfer derived the following generalized expression for the moisture diffusivity in capillary-porous bodies:

$$\bar{q}_m = -\lambda_m \operatorname{grad} \Phi - \frac{\lambda_m}{c_m \gamma_m w_r^2} \frac{\partial \bar{q}_m}{\partial \tau}. \tag{1}$$

If it is recalled that the rate of capillary movement of liquid w_r and the relaxation time in hours τ_r are related to the moisture diffusion coefficient $a_m = \lambda_m / c_m \gamma_m$ by the expression $w_r^2 = a_m/\tau_r$, then Eq. (1) can be rewritten in the form

$$\bar{q}_m = -\lambda_m \operatorname{grad} \Phi - \tau_r \partial \bar{q}_m / \partial \tau. \tag{2}$$

In (1) and (2) $\Phi(x, y, z, \tau)$ is a function describing the moisture distribution in the body, and $\bar{q}_m$ is the liquid flux density.

Equations (1) and (2) were derived by A. V. Luikov starting from the general principles of the thermodynamics of irreversible processes.

According to (1), the liquid flux $\bar{q}_m$ is the sum of two fluxes: that caused by the presence of a moisture content gradient and that determined by molar movement of liquid along the capillaries.

Equation (1) describes all the intermediate processes lying between the two limit cases

$$\bar{q}_m = -\lambda_m \operatorname{grad} \Phi$$

and

$$\bar{q}_m = -\tau_r \partial \bar{q}_m / \partial \tau.$$

In the first of these cases the liquid flux is determined only by the molecular process of moisture diffusivity (diffusion of liquid) under the action of a moisture content gradient. This is the case usually examined in the theory of moisture diffusivity. In this case the rate of propagation of moisture is assumed infinitely great. If the initial moisture content is zero and an instantaneous source of moisture commences to act, then, starting from some time $\tau > 0$, at any point the moisture content of the body is different from zero, i.e., the action of the moisture source is propagated instantaneously. In fact, as shown by observation, propagation of moisture in materials exposed to wetting or drying takes place at a certain definite finite rate.

This is particularly true of materials having a porous structure. Therefore, in analyzing such cases it is necessary to proceed from (1). In the second limit case the moisture flux depends only on macroscopic displacement under the action of capillary forces. From the relation for this case it follows that $\bar{q}_m = \bar{q}_0 \exp(-\tau/\tau_r)$, i.e., the liquid flux decreases exponentially with time from a certain initial value $\bar{q}_0$.

Using (1) or (2), we can obtain a generalized differential equation of moisture diffusivity

$$\frac{\partial^2 \Phi}{\partial \tau^2} + \frac{w_r^2}{a_m} \frac{\partial \Phi}{\partial \tau} = w_r^2 \nabla^2 \Phi, \tag{3}$$

or

$$\frac{\partial^2 \Phi}{\partial \tau^2} + \frac{1}{\tau_r} \frac{\partial \Phi}{\partial \tau} = \frac{a_m}{\tau_r} \nabla^2 \Phi. \tag{4}$$

Introducing the notation $2h = 1/\tau_r$ and $a^2 = a_m/\tau_r$, we obtain

$$\frac{\partial^2 \Phi}{\partial \tau^2} + 2h \frac{\partial \Phi}{\partial \tau} = a^2 \nabla^2 \Phi. \tag{5}$$

Below we give the solution for two one-dimensional boundary value problems for the equation of moisture diffusivity (5).

I. We will find the solution of the boundary value problem for the hyperbolic equation of moisture diffusivity with boundary conditions of the first and second kind:

$$\frac{\partial^2 \Phi}{\partial \tau^2} + 2h \frac{\partial \Phi}{\partial \tau} = a^2 \frac{\partial^2 \Phi}{\partial x^2} \quad (0 \leqslant x \leqslant R), \tag{6}$$

$$\Phi(x,\ 0) = \Phi_0 = \text{const}, \tag{7}$$

$$\frac{\partial \Phi(x,\ 0)}{\partial \tau} = 0, \tag{8}$$

$$\Phi(0,\ \tau) = \Phi_n = \text{const}, \tag{9}$$

$$\frac{\partial \Phi(R,\ \tau)}{\partial x} = 0. \tag{10}$$

Since boundary condition (9) is inhomogeneous, the solution of problem (6)–(10) will be sought in the form of a sum $\Phi = u^* + v$, where u^* is a function satisfying the boundary conditions (9) and (10).

The function u^* can be taken in the form

$$u^* = \Phi_n \left(\frac{x^2}{2R^2} + \frac{R-x}{R} \right). \tag{11}$$

For determination of the function v it is necessary to solve the inhomogeneous equation

$$\frac{\partial^2 v}{\partial \tau^2} + 2h \frac{\partial v}{\partial \tau} = a^2 \frac{\partial^2 v}{\partial x^2} + a^2 \frac{\Phi_n}{R^2} \tag{12}$$

with initial conditions

$$v(x,\ 0) = \Phi_0 - u^*, \tag{13}$$

$$\partial v(x,\ 0)/\partial \tau = 0 \tag{14}$$

and boundary conditions

$$v(0,\ \tau) = 0, \tag{15}$$

$$\partial v(R,\ \tau)/\partial x = 0. \tag{16}$$

We will seek a solution of problem (12)–(16) in the form of an expansion in eigenfunctions of the corresponding homogeneous problem

$$v(x,\ \tau) = \sum_{n=1}^{\infty} T_n(\tau) X_n(x) \quad (0 \leqslant x \leqslant R). \tag{17}$$

Functions $X_n(x)$ and $T_n(\tau)$ are solutions of the following equations:

$$X_n'' + kX_n = 0, \quad k > 0, \tag{18}$$

$$T_n'' + 2hT_n' + ka^2 T_n = -2a^2 \Phi_n/\mu_n R^2. \tag{19}$$

The solution of (18), taking account of boundary conditions (15) and (16), has the form

$$X_n(x) = \sin \mu_n \frac{x}{R}, \tag{20}$$

where

$$\mu_n = \sqrt{k_n}\, R = (2n-1)\pi/2 \quad (n = 1,\ 2,\ \ldots). \tag{21}$$

The solution of (19) when $a^2 \mu_n^2/R^2 > h^2$ has the form

$$T_n(\tau) = -2\Phi_n/\mu_n^3 + \exp(-h\tau)[A_n \cos q_n \tau + B_n \sin q_n \tau], \tag{22}$$

where $q_n = \sqrt{a^2 \mu_n^2/R^2 - h^2}$.

Taking into account (20) and (22), we rewrite (17) thus:

$$v = \sum_{n=1}^{\infty} \sin \mu_n \frac{x}{R} \left\{ (A_n \cos q_n \tau + B_n \sin q_n \tau) \exp(-h\tau) - \frac{2\Phi_n}{\mu_n^3} \right\}. \tag{23}$$

The arbitrary constants A_n and B_n are determined from the initial conditions (13) and (14).

Computations give the following values for A_n and B_n:

$$A_n = \frac{2\Phi_n}{\mu_n^3} + \frac{2}{R} \int_0^R (\Phi_0 - u^*) \sin \mu_n \frac{x}{R}\, dx, \tag{24}$$

$$B_n = \frac{h}{q_n} A_n. \tag{25}$$

The coefficient A_n can be written differently. Substituting the value of the function u^* in the integral in (24) and carrying out the integration, we find

$$A_n = \frac{2}{\mu_n} \left[(\Phi_0 - \Phi_n) + \frac{2\Phi_n}{\mu_n^2} \right]. \tag{26}$$

Taking into consideration that $\Phi(x, \tau) = u^* + v(x, \tau)$, and using (23), (25), and (26), after a number of transformations we obtain the final solution of the initial problem for $a^2 \mu_n^2 / R^2 > h^2$ in the form

$$\Phi(x, \tau) = \Phi_n \left(\frac{x}{R} - 1 \right)^2 + \exp(-h\tau) \sum_{n=1}^{\infty} \frac{2}{\mu_n} \left[(\Phi_0 - \Phi_n) + \frac{2\Phi_n}{\mu_n^2} \right] \times$$
$$\times \left(\cos q_n \tau + \frac{h}{q_n} \sin q_n \tau \right) \sin \mu_n \frac{x}{R}. \tag{27}$$

II. We will also examine the following boundary value problem for the hyperbolic equation of moisture diffusivity with boundary conditions of the second and third kind:

$$\frac{\partial^2 \Phi}{\partial \tau^2} + 2h \frac{\partial \Phi}{\partial \tau} = a^2 \frac{\partial^2 \Phi}{\partial x^2} \quad (0 \leqslant x \leqslant R), \tag{28}$$

$$\Phi(x, 0) = \varphi_1(x), \tag{29}$$

$$\frac{\partial \Phi(x, 0)}{\partial \tau} = \varphi_2(x), \tag{30}$$

$$\frac{\partial \Phi(0, \tau)}{\partial x} = 0, \tag{31}$$

$$-\lambda_m \frac{\partial \Phi(R, \tau)}{\partial x} + \alpha_m [\Phi_c - \Phi(R, \tau)] = 0, \tag{32}$$

$$\Phi_c = \text{const}.$$

The solution of this problem takes the form

$$\Phi(x, \tau) = \Phi_c - \exp(-h\tau) \sum_{n=1}^{\infty} (A_n \cos q_n \tau + B_n \sin q_n \tau) \cos \mu_n \frac{x}{R}, \tag{33}$$

$$A_n = \frac{\mu_n}{\mu_n + \sin \mu_n \cos \mu_n} \frac{2}{R} \int_0^R f_1(x) \cos \mu_n \frac{x}{R}\, dx, \tag{34}$$

$$B_n = \frac{h}{q_n} A_n + \frac{\mu_n}{q_n(\mu_n + \sin\mu_n \cos\mu_n)} \frac{2}{R} \int_0^R f_2(x) \cos\mu_n \frac{x}{R} dx, \tag{35}$$

$$f_1(x) = \Phi_c - \varphi_1(x), \quad f_2(x) = -\varphi_2(x), \tag{36}$$

$$q_n = \sqrt{\frac{\mu_n^2 a^2}{R^2} - h^2}, \quad \frac{\mu_n^2 a^2}{R^2} > h^2. \tag{37}$$

The numbers μ_n are the roots of the characteristic equation

$$\operatorname{ctg}\mu = \frac{\mu}{\mathrm{Bi}}, \tag{38}$$

where

$$\mathrm{Bi} = \frac{\alpha_m}{\lambda_m} R.$$

In conclusion, we point out that the solutions given above can be used not only to determine the moisture content field and the drying rate for various porous materials, but also to calculate the relaxation time τ_r and the rate of moisture transfer w_r.

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